Estimating / computing $c(G)$, $m(G(\mathbb{R}))$

Main result so far

Thm $G/\mathbb{Q}$ ss group $\Gamma \le G(\mathbb{R})$ arith

$$j^q : I_G^\Gamma \xrightarrow{\sim} H^q(\Gamma)$$

for $q \le \min(c(G), m(G(\mathbb{R})))$ $\qquad \lfloor x \rfloor' = \min\{n \in \mathbb{Z} \mid n < x\}$

Today:

Thm G almost simple

$$\min(c(G), m(G(\mathbb{R})) \ge \left\lfloor \frac{\operatorname{rk}_{\mathbb{Q}}(G)}{4} \right\rfloor'$$

Ex: $G = \mathrm{Res}_{F/\mathbb{Q}} SL_n$

$$c(G) = \left\lfloor \frac{[F:\mathbb{Q}](n-1)}{2} \right\rfloor'$$

$$\begin{aligned} m(SL_n(\mathbb{R})) &\geq \left\lfloor \frac{n+2}{4} \right\rfloor' \\ m(SL_n(\mathbb{C})) &= \left\lfloor \frac{n}{2} \right\rfloor' \end{aligned} \Rightarrow m(G(\mathbb{R})) \geq \begin{cases} \left\lfloor \frac{n+2}{4} \right\rfloor' & F \text{ has a real place} \\ \left\lfloor \frac{n}{2} \right\rfloor' & \text{otherwise} \end{cases}$$

The Constant $c(G)$

Φ irreducible root system (possibly non reduced, i.e. $\alpha, 2\alpha \in \Phi$)

$\cup\!|$

Δ basis of simple roots

For $v = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \mathbb{R}\Phi$ write

- $v \geq 0$ if $c_\alpha \geq 0$ $\forall \alpha \in \Delta$
- $v \gg 0$ if $c_\alpha > 0$ " "

Let

- $r = \frac{1}{2} \sum_{\substack{\alpha \in \Phi \\ \alpha \geq 0}} \alpha$
- α_0 longest root ($\alpha_0 \geq \alpha$ $\forall \alpha \in \Phi$)

<u>def</u>: $c(\Phi) = \max\{q \in \mathbb{Z} \mid r - q\alpha_0 \gg 0\}$

<u>Ex</u>: A_n

$$\Phi = \{e_i - e_j \mid 1 \leq i,j \leq n+1 \quad i \neq j\}$$

$$\rotatebox{90}{$\subseteq$}$$

$$\Delta = \{\underbrace{e_i - e_{i+1}}_{\alpha_i} \mid i = 1, \dots, n\}$$

$$\alpha_0 = e_1 - e_{n+1} = \alpha_1 + \dots + \alpha_n$$

$$r = \frac{n}{2}\alpha_1 + \frac{(n-1)2}{2}\alpha_2 + \frac{(n-2)3}{2}\alpha_3 + \dots + \frac{n}{2}\alpha_n$$

$$\Rightarrow c(\Phi) = \lfloor \tfrac{n}{2} \rfloor'$$

<u>prop</u>: For any irreducible Φ $\quad c(\Phi) \geq \lfloor \frac{\#\Delta}{2} \rfloor'$

<u>pf</u>: case by case

$$\boxed{c(\Phi_1 \sqcup \Phi_2 \sqcup \dots) = \min c(\Phi_i)}$$

Now $G/\mathbb{Q}$ semisimple group

- $S \subseteq P \subseteq G$ $\quad S$ max split torus, P minimal parabolic
- $U \subseteq P$ unip radical
- $\Phi(G/\mathbb{Q})$ relative root system

$\mathcal{U} = \{$ nonzero characters of S occuring in $\mathrm{Lie}(G)\}$

$\Phi^+(G/\mathbb{Q}) = \{$ " " in $\mathrm{Lie}(V)\}$

Recall: $c(G) = \max\{q \in \mathbb{Z} \mid \rho - \lambda > 0 \ \forall \lambda$ occuring in $\bigoplus_{i=1}^{q} \Lambda^i \mathrm{Lie}(U)\}$

<u>Prop</u>: $c(G) \geq c(\Phi(G/\mathbb{Q}))$ $\left(\geq \left\lfloor \frac{\mathrm{rank}_{\mathbb{Q}}(G)}{2} \right\rfloor \text{ if } G \text{ almost simple} \right)$

<u>Pf</u>: We may assume G is almost simple ($\Leftrightarrow \Phi(G/\mathbb{Q})$ is irreducible)

- $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} m_\alpha \cdot \alpha \geq \frac{1}{2} \sum_{\alpha \in \Phi^+} 1 \cdot \alpha = r$

m_α: multiplicity of α in $\mathrm{Lie}(U)$

- Any weight of $\mathrm{Lie}(V)$ is $\leq d_0$

$\Rightarrow$ Any weight of $\Lambda^i \mathrm{Lie}(V)$ is $\leq i d_0$ $\square$

<u>Rem</u>: • If $G = \mathrm{Res}_{F/\mathbb{Q}} G'$ $\Phi(G/\mathbb{Q}) = \Phi(G'/F)$ but $\rho \geq [F:\mathbb{Q}] \cdot r$

$\Rightarrow c(G) \geq [F:\mathbb{Q}] \cdot c(\Phi(G/\mathbb{Q}))$

• $G = \mathrm{SL}_n/\mathbb{Q}$ we do have $=$

The Constant $m(G(\mathbb{R}))$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ semisimple noncompact real Lie algebra w/ Cartan decomposition

$$m(\mathfrak{g}) = \max\{q \in \mathbb{Z} \mid H^q_{\mathfrak{g}} \text{ is positive definite}\}$$

where: • $H_g^q(\xi) = \frac{A}{q}\langle \xi, \xi \rangle + \langle Q(\xi), \xi \rangle \quad \xi \in \mathrm{Sym}^2 \mathfrak{p}$

• $\langle\, , \rangle$ is induced by B_g on $\mathfrak{p}$

• A is a constant relating B_g and B_k on k

$A := \min\{ 1 + B_k(X, X) \mid X \in k \;\; B_g(X, X) = -1 \}$

$(0 < A \leq 1)$

• Q is curvature matrix: $Q \circlearrowleft \mathrm{Sym}^2 \mathfrak{p} \subseteq \mathfrak{p} \otimes \mathfrak{p}$

$$\langle Q(X \otimes Y), Z \otimes W \rangle = B_g([Y, W], [Z, X])$$
$$\| $$
$$\big(B_g([[Y, W], Z], X) \big)$$

$\underbrace{\qquad\qquad}_{R}$ $\qquad {}^{a}Q = R_{ab}{}^{cd}$ "

$R^{a}{}_{bcd}$

- Q preserves $\mathrm{Sym}^2 \mathfrak{p}$
- Q is self adjoint $\langle Q\xi, \eta\rangle = \langle \xi, Q\eta\rangle$
- Q commutes with $K \circlearrowright \mathrm{Sym}^2 \mathfrak{p}$

So if $-N$ is the smallest eigenvalue of Q

then $\displaystyle \min_{\xi \in \mathrm{Sym}^2 \mathfrak{p}} \frac{\langle Q\xi, \xi\rangle}{\langle \xi, \xi\rangle} = -N$

So H^{a}_{g} is $>0 \iff \frac{A}{a} - N > 0 \Rightarrow m(g) = \left\lfloor \frac{A}{N} \right\rfloor'$

Computation of A:

$\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$ orthogonal for both $B_{\mathfrak{g}}, B_{\mathfrak{k}}$

simple ideals

$B_{\mathfrak{k}} = 0$ on $\mathfrak{z}(\mathfrak{k})$ and $B_{\mathfrak{k}} = a_i B_{\mathfrak{g}}$ on $\mathfrak{k}_i$

$A = \min_i (1 - a_i)$ (or 1 if $\mathfrak{k}$ is abelian)

Shortcut (Matsushima)

$$\dim \mathfrak{z}(\mathfrak{k}) + \sum_i (1 - a_i) \cdot \dim \mathfrak{k}_i = \frac{1}{2} \dim \mathfrak{p}$$

So if $\mathfrak{k}$ is simple

$$A = (1-a_1) = \frac{\dim \mathfrak{p}}{2\dim \mathfrak{k}}$$

<u>EX</u>: $\mathfrak{sl}_n(\mathbb{C})$ $\quad A = \frac{1}{2}$

$\mathfrak{sl}_n(\mathbb{R})$ $\quad A = \frac{n+2}{2n}$ $\quad$ (when $n=4$ $\mathfrak{k}$ isn't simple but $a_1 = a_2$)

Computation of N (after Kaneyuki, Nagano)

<u>Strategy</u>:

1. Bound N for $\mathfrak{g}$ in terms of N for $\mathfrak{g}_{\mathbb{C}}$
2. Compute N case by case for complex simple Lie algebras

Step 1:

We consider some other Lie algebras

- $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ Compact dual
- $\mathfrak{g}_u \times \mathfrak{g}_u = \sigma(\mathfrak{g}_u) \oplus \tau(\mathfrak{g}_u)$ (compact dual of $\mathfrak{g}_{\mathbb{C}}$

$$\sigma, \tau : \mathfrak{g}_u \to \mathfrak{g}_u \times \mathfrak{g}_u$$

$$\sigma(X) = (X, X)$$

$$\tau(X) = \frac{1}{\sqrt{2}}(X, -X)$$ ⟸ τ isometric for killing forms

⇝ can define curvature matrices

$$Q_{(\mathfrak{g}_u, \mathfrak{k})} \qquad Q_u = Q_{(\mathfrak{g}_u \times \mathfrak{g}_u, \sigma(\mathfrak{g}_u))}$$

under $\mathrm{Sym}^2 P \simeq \mathrm{Sym}^2 iP \quad Q = -Q_{(g_v,k)}$

so N largest e-val of $Q_{(g_v,k)}$

Let N_v be largest evl of Q_v

$$\boxed{m = \lfloor \tfrac{A}{N} \rfloor'}$$

<u>Prop</u>: $N \leq 2N_v$

<u>pf</u>: Pick $\xi \in \mathrm{Sym}^2 P$ w/ $Q_{(g_v,k)}\xi = N\xi$

$$\begin{aligned} 2N_v \langle \xi,\xi\rangle_{g_v} &= 2N_v \langle \mathrm{Sym}^2\tau(\xi), \mathrm{Sym}^2\tau(\xi)\rangle_v \\ &\geq 2\langle Q_v(\mathrm{Sym}^2\tau(\xi)), \mathrm{Sym}^2\tau(\xi)\rangle \\ &\overset{(*)}{=} \langle Q_{(g_v,k)}(\xi), \xi\rangle_{(g_v,k)} \\ &= N\langle \xi,\xi\rangle \end{aligned}$$

(*) follows from

$$\Pi \circ Q_v \circ \mathrm{Sym}^2\tau = \frac{1}{2}\,\mathrm{Sym}^2\tau \circ Q_{(\mathfrak{g}_v, k)}$$

where $\Pi : \mathrm{Sym}^2(\tau)(\mathrm{Sym}^2 \mathfrak{g}_v) \to \mathrm{Sym}^2(\tau)(\mathrm{Sym}^2(\square))$

is orthogonal projection

<u>Prop</u> Let $X \in \mathfrak{g}_v$ be root vector for highest root. Then $\tau(X)^2$ is eigenvector for Q_v w/ eigenvalue N_v

<u>Rmk</u>: Easy: $\tau(X)^2$ is eigenvector and compute evalue

Hard: It is the largest eigenvalue

Ex: Type A_{n-1} one compute $N_v = \frac{1}{n}$

- $m(\mathfrak{sl}_n(\mathbb{C})) = \left\lfloor \frac{\frac{1}{2}}{\frac{1}{n}} \right\rfloor' = \left\lfloor \frac{n}{2} \right\rfloor'$
- $\mathfrak{sl}_n(\mathbb{R})$ $N \leq 2N_v = \frac{2}{n}$

$$m(\mathfrak{sl}_n(\mathbb{R})) \geq \left\lfloor \frac{\frac{n+2}{2n}}{\frac{2}{n}} \right\rfloor = \left\lfloor \frac{n+2}{4} \right\rfloor'$$

Strategy for computing all eigenvalues of Q_v:

- decompose $\mathrm{Sym}^2 \mathfrak{g}_v = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots$ as a $\mathfrak{g}$ rep

(≤ 4 factors, no multiplicities)

- Q_ω commutes w/ $\mathfrak{g}_\omega \curvearrowright \mathrm{Sym}^2\mathfrak{g}_\omega$

  $\rightsquigarrow$ Q_ω acts by scalar N_i on V_{λ_i}

- For any weight λ, we can find a basis for $(\mathrm{Sym}^2\mathfrak{g}_\omega)_\lambda$ and use that to compute (painfully!)

  $$\mathrm{tr}(Q_\omega | (\mathrm{Sym}^2\mathfrak{g}_\omega)_\lambda) = \sum_i m_\lambda(V_{\lambda_i}) \cdot N_i$$

  $\rightsquigarrow$ inductive strategy for computing N_i

Shortcuts:
- easy to compute $N_i \Leftrightarrow T(X^2) \in V_{\lambda_i}$
- always have $V_0 \Longleftrightarrow$ killing form $N = -\frac{1}{2}$
- Compute $\mathrm{tr}\, Q$

$\underline{\text{Ex}}$: A_n $\quad \mathfrak{g} = V_{\pi_1+\pi_n}$ $\quad \mathrm{Sym}^2 V_{\pi_1+\pi_n} = V_{2\pi_1+2\pi_n} + V_{\underline{\pi_2+\pi_{n-1}}} + V_{\pi_1+\pi_n} + V_0$

$$\frac{1}{2(n+1)}, \quad \underline{\frac{-1}{2(n+1)}}, \quad \frac{-1}{2}, \quad \frac{-1}{2}$$

$\underline{\mathfrak{k} \hookrightarrow \mathrm{Sym}^2 \mathfrak{p}}$ $\qquad \mathfrak{g} \hookrightarrow \mathrm{Sym}^2 \mathfrak{g}$

$\mathrm{rk}_{\mathbb{Q}} G$ small

$SO(n,1)$ $\qquad \mathrm{rk}_{\mathbb{R}} G$ big

$m(G(\mathbb{R}))$ big

$G/\mathbb{Q}$ $\qquad \underline{G(\mathbb{R})} = \prod \underline{H_i}$